Kepler's Law

By

Dr.Vinod Kumar Singh

Associate Professor Physics K .S. Saket P G College Ayodhya

Email: vks423272@gmail.com

Motion is always relative. Based on the energy of the particle under motion, the motions are classified into two types:

- **Bounded Motion**
- **Unbounded Motion**

In bounded motion, the particle has negative total energy (E<0) and has two or more extreme points where the total energy is always equal to the potential energy of the particle i.e the kinetic energy of the particle becomes zero.

For eccentricity $0 \leq e < 1$, $E<0$ implies the body has bounded motion. A circular orbit has eccentricity $e = 0$ and elliptical orbit has eccentricity $e < 1$.

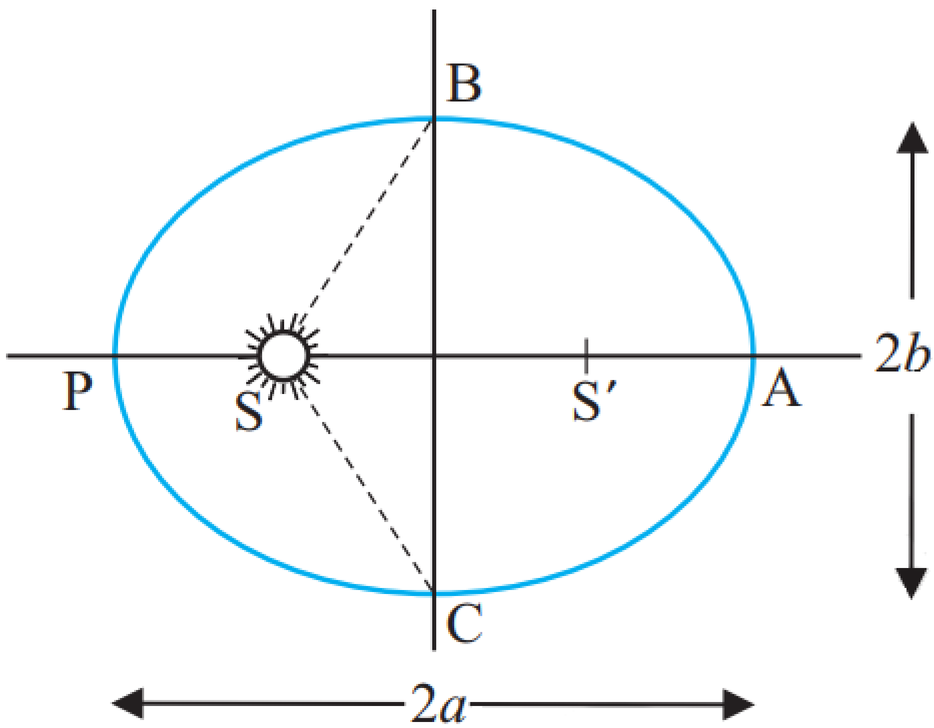
In unbounded motion, the particle has positive total energy (E>0) and has a single extreme point where the total energy is always equal to the potential energy of the particle i.e the kinetic energy of the particle becomes zero.

For eccentricity $e \geq 1$, $E > 0$ implies the body has unbounded motion. Parabolic orbit has eccentricity $e = 1$ and Hyperbolic path has eccentricity $e>1$.

Kepler First law – The Law of Orbits

According to Kepler's first law," All the planets revolve around the sun in elliptical orbits having the sun at one of the foci". The point at which the planet is close to the sun is known as perihelion and the point at which the planet is farther from the sun is known as aphelion.

It is the characteristics of an ellipse that the sum of the distances of any planet from two foci is constant. The elliptical orbit of a planet is responsible for the occurrence of seasons.



Kepler First Law – The Law of Orbits

Motion underInverse Square Force – Kepler's Problem: Let us consider the motion of a particle in a central force field. The particle may be a star interacting gravitationally with another star or it may be a planet moving in gravitational field of sun or it may be an electron moving in the coulomb field of the nucleus.

The inverse square central force is given as –

$f(r) = -\frac{k}{r^2}$ k – any constant

The equation of orbit is given as –

$$\frac{\partial^2 u}{\partial \theta^2} + u = -\frac{m}{l^2 u^2} f\left(\frac{1}{u}\right) \qquad \text{-------------- (1)}$$

Putting $f\left(\frac{1}{u}\right) = f(r) = -\frac{k}{r^2} = -k\,u^2$, we get – $\frac{\partial^2 u}{\partial \theta^2} + u = \frac{m\,k}{l^2}$ -------------- (2)

Let – $y = u - \frac{mk}{l^2}$ $\Rightarrow$ $\frac{\partial^2 y}{\partial \theta^2} = \frac{\partial^2 u}{\partial \theta^2}$ so equation (2) becomes –

$$\frac{\partial^2 u}{\partial \theta^2} + y = 0 \qquad \text{-------------- (3)}$$

The general solution of equation (3) is –

$$y = u' \cos(\theta - \theta') \quad \Rightarrow \quad u = \frac{mk}{l^2} + u' \cos(\theta - \theta') \quad \Rightarrow \quad \frac{1}{r} = \frac{mk}{l^2} + u' \cos(\theta - \theta')$$

where u' and θ' are constants. For simplicity, if we orient our co- ordinate system so that $\theta' = 0$ then –

$$\frac{1}{r} = \frac{mk}{l^2} + u' \cos\theta \quad \Rightarrow \quad r = \frac{1}{\frac{mk}{l^2} + u' \cos\theta} = \frac{l^2}{mk + u' l^2 \cos\theta}$$

$$\boxed{r = \frac{l^2/mk}{1 + \frac{u' l^2}{mk}\cos\theta}} \qquad \text{-------------- (4)}$$

θ' can be recognized as one of the angles at which the orbit turns because $\theta = \theta'$ or $\theta = \theta' + \pi$ results in a maximum and minimum value for r. Thus θ' can be called a turning point.

Equation (4) resembles with the standard equation of conic section –

$$\boxed{r = \frac{P}{1+\varepsilon\cos\theta}}$$ ------------- (5)

Comparing equation (4) and (5), we get –

$$P = \frac{l^2}{mk} \qquad \text{and} \qquad \varepsilon = \frac{u' l^2}{mk}$$

Hence orbit under inverse square force is always a conic section. ε is called 'Eccentricity of the orbit'. This coincides with the Kepler's first law of planetary motion which states that the orbits are conic sections (ellipses for the planets) with the centre of force at one of the foci.

<u>Ques.1:</u> A particle describes a circular orbit given by r= 2a cosθ under the influence of an attractive central force directed towards a fixed point on the circle. Show that the force varies as the inverse 5th power of the distance.

OR

The orbit described by a particle under central force is r= 2a cos θ. obtain the functional form of f(r).

Solution: In polar co-ordinates, equation of a circle of radius 'a' passing through the origin is – $r = 2a\cos\theta$

putting $r = \frac{1}{u}$ we get –

$$\frac{1}{u} = 2a\cos\theta \Rightarrow u = \frac{1}{2a\cos\theta} \quad \Rightarrow \quad u = \frac{\operatorname{Sec}\theta}{2a}$$ ------------- (1)

$$\frac{\partial u}{\partial\theta} = \frac{1}{2a}\operatorname{Sec}\theta\tan\theta$$

$$\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{2a}\left[\operatorname{Sec}\theta\tan\theta.\tan\theta + \operatorname{Sec}\theta\operatorname{Sec}^2\theta\right]$$ ------------- (2)

The differential equation of orbit is given by –

$$-\frac{l^2u^2}{m}\left[\frac{\partial^2 u}{\partial\theta^2}+u\right]=f\left(\frac{1}{u}\right) \qquad \text{------------- (3)}$$

Putting values of $\frac{\partial^2 u}{\partial\theta^2}$ and u in equation (3), we get –

$$f\left(\frac{1}{u}\right)=-\frac{l^2u^2}{m}\left[\frac{1}{2a}\left(\text{Sec}\,\theta\tan^2\theta+\text{Sec}^3\theta+\text{Sec}\theta\right)\right]=-\frac{l^2u^2}{m}\left[\frac{1}{2a}\left[\text{Sec}\,\theta\left\{1+\tan^2\theta\right\}+\text{Sec}^3\theta\right]\right]$$

$$f\left(\frac{1}{u}\right)=\frac{l^2u^2}{m}\left[\frac{1}{2a}2\,\text{Sec}^3\theta\right]-=-\frac{l^2u^2}{am}\text{Sec}^3\theta$$

from equation (1) Sec θ = 2au

$$\therefore f\left(\frac{1}{u}\right)=-\frac{l^2u^2}{am}8a^3u^3=-\frac{8a^2l^2}{m}u^5=-\frac{8a^2l^2}{m}\frac{1}{r^5} \qquad \textbf{OR} \qquad f(r)\propto\frac{1}{r^5}$$

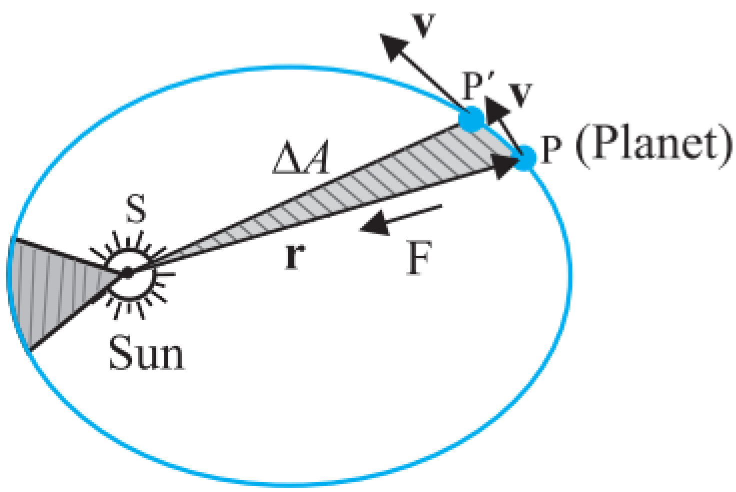
i.e. the force varies as inverse 5th power of the distance.

Kepler's Second Law – The Law of Equal Areas

Kepler's second law states " The radius vector drawn from the sun to the planet sweeps out equal areas in equal intervals of time"

As the orbit is not circular, the planet's kinetic energy is not constant in its path. It has more kinetic energy near perihelion and less kinetic energy near aphelion implies more speed at perihelion and less speed (v_{min}) at aphelion. If r is the distance of planet from sun, at perihelion (r_{min}) and at aphelion (r_{max}), then,

$r_{min} + r_{max} = 2a \times$ (length of major axis of an ellipse) . . . . (1)



Kepler's Second Law – The law of Equal Areas

For an infinitesimal movement of the planet in a time interval in an elliptical orbit, the area swept by the planet in time is given by;

dA/dt = d/dt [1/2 × r × (v dt)]= 1/2 × rv . . . . . (2)

At perihelion $r = r_{min}$, $v = v_{max}$ then from Equation 2;

dA/dt = $1/2 \times r_{min} \times v_{max})$ = $[m \times v_{max} \times r_{min}]/2m$ = L/2m;

At aphelion $r = r_{max}$, $v = v_{min}$ then from Equation 2;

dA/dt = $1/2 \times v_{min} \times r_{max}$ = $[m \times v_{min} \times r_{max}]/2m$ = L/2m

Kepler's second law can also be stated as "The areal velocity of a planet revolving around the sun in elliptical orbit remains constant which implies the angular momentum of a planet remains constant". As the angular momentum is constant all planetary motions are planar motions, which is a direct consequence of central force.

Kepler's Third Law – The Law of Periods

According to Kepler's law of periods," The square of the time period of revolution of a planet around the sun in an elliptical orbit is directly proportional to the cube of its semi-major axis".

$\mathbf{T^2 \propto a^3}$

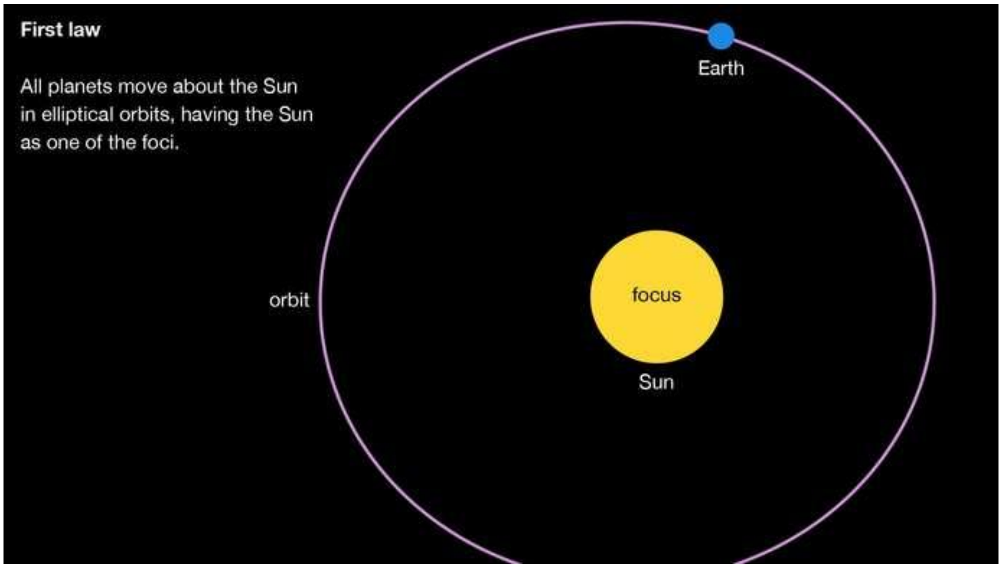
Shorter the orbit of the planet around the sun, shorter the time taken to complete one revolution. Using the equations of Newton's law of gravitation and laws of motion, Kepler's third law takes a more general form:

$P^2 = 4\pi^2 / [G(M_1 + M_2)] \times a^3$

where M_1 and M_2 are the masses of the two orbiting objects in solar masses.

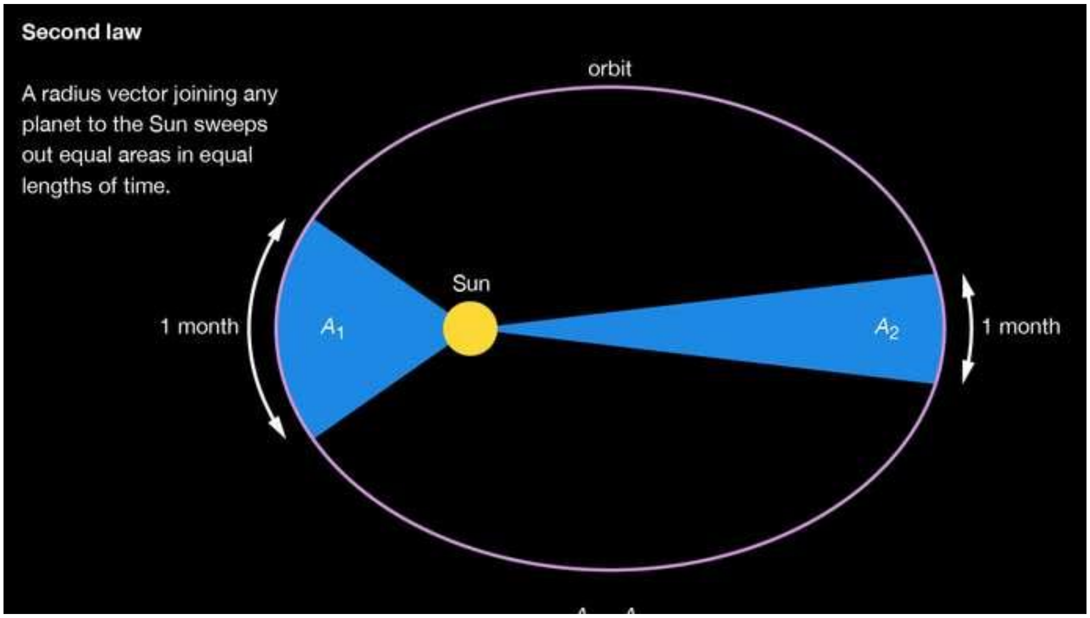
Kepler's laws of planetary motion

Kepler's laws of planetary motion, in astronomy and classical physics, laws describing the motions of the planets in the solar system. They were derived by the German astronomer Johannes Kepler, whose analysis of the observations of the 16th-century Danish astronomer Tycho Brahe enabled him to announce his first two laws in the year 1609 and a third law nearly a decade later, in 1618. Kepler himself never numbered these laws or specially distinguished them from his other discoveries.

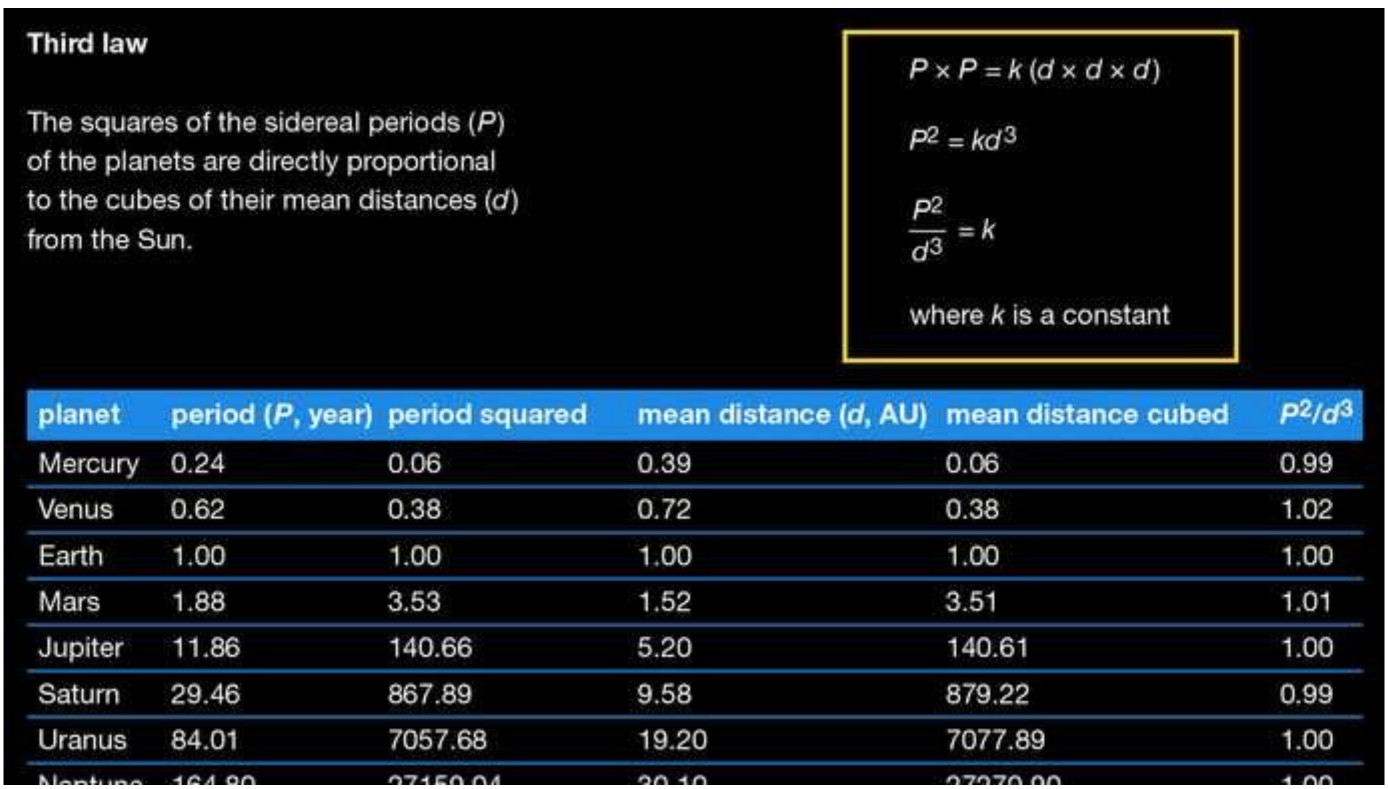


Kepler's first law

Kepler’s three laws of planetary motion can be stated as follows: (1) All planets move about the Sun in elliptical orbits, having the Sun as one of the foci. (2) A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time. (3) The squares of the sidereal periods (of revolution) of the planets are directly proportional to the cubes of their mean distances from the Sun. Knowledge of these laws, especially the second (the law of areas), proved crucial to Sir Isaac Newton in 1684–85, when he formulated his famous law of gravitation between Earth and the Moon and between the Sun and the planets, postulated by him to have validity for all objects anywhere in the universe. Newton showed that the motion of bodies subject to central gravitational force need not always follow the elliptical orbits specified by the first law of Kepler but can take paths defined by other, open conic curves; the motion can be in parabolic or hyperbolic orbits, depending on the total energy of the body. Thus, an object of sufficient energy—e.g., a comet—can enter the solar system and leave again without returning. From Kepler’s second law, it may be observed further that the angular momentum of any planet about an axis through the Sun and perpendicular to the orbital plane is also unchanging.



Kepler's second law



planet	period (P, year)	period squared	mean distance (d, AU)	mean distance cubed	P^2/d^3
Mercury	0.24	0.06	0.39	0.06	0.99
Venus	0.62	0.38	0.72	0.38	1.02
Earth	1.00	1.00	1.00	1.00	1.00
Mars	1.88	3.53	1.52	3.51	1.01
Jupiter	11.86	140.66	5.20	140.61	1.00
Saturn	29.46	867.89	9.58	879.22	0.99
Uranus	84.01	7057.68	19.20	7077.89	1.00

Kepler's third law

Kepler's third law of planetary motion. The squares of the sidereal periods (*P*) of the planets are directly proportional to the cubes of their mean distances (*d*) from the Sun.

The usefulness of Kepler's laws extends to the motions of natural and artificial satellites, as well as to stellar systems and extrasolar planets. As formulated by Kepler, the laws do not, of course, take into account the gravitational interactions (as perturbing effects) of the various planets on each other. The general problem of accurately predicting the motions of more than two bodies under their mutual attractions is quite complicated; analytical solutions of the three-body problem are unobtainable except for some special cases. It may be noted that Kepler's laws apply not only to gravitational but also to all other inverse-square-law forces and, if due allowance is made for relativistic and quantum effects, to the electromagnetic forces within the atom.